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NUMERICAL SOLUTION OF A SINGULAR INTEGRAL EQUATION WITH THE HILBERT KERNEL BY THE METHOD OF DISCRETE SINGULARITIES

In the paper the method of discrete singularities is used for constructing a discrete mathematical model of a first kind singular integral equation with the Hilbert kernel in the case when the auxiliary conditions introduced to ensure the uniqueness of solution to the equation are given by functionals. The existence of the unique solution to the discrete model is proved and the rate of convergence of the solution of the discrete problem to the exact solution of the initial singular integral equation is estimated under some smoothness assumptions.

Key words: singular integral equations, method of discrete singularities, Hilbert kernel, discrete model.

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ЧИСЕЛЬНЕ РОЗВ'ЯЗАННЯ МЕТОДОМ ДИСКРЕТНИХ ОСОБЛИВОСТЕЙ ОДНОГО СІНГУЛЯРНОГО ІНТЕГРАЛЬНОГО РІВНЯННЯ З ЯДРОМ ГІЛЬБЕРТА

На основі методу дискретних особливостей побудовано дискретну математичну модель сингулярного інтегрального рівняння першого роду з ядром Гілберта у випадку, коли додаткова умова, що дозволяє отримати єдиний розв'язок цього рівняння, є функціонал. Доведена однозначна розв'язність дискретної моделі і дана оцінка швидкості збіжності розв'язку дискретної задачі до точного розв'язку сингулярного інтегрального рівняння при деяких припущеннях гладкості.

Ключові слова: сингулярне інтегральне рівняння, метод дискретних особливостей, ядро Гілберта, дискретна модель.

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ЧИСЛЕННОЕ РЕШЕНИЕ МЕТОДОМ ДИСКРЕТНЫХ ОСОБЕННОСТЕЙ ОДНОГО СИНГУЛЯРНОГО ИНТЕГРАЛЬНОГО УРАВНЕНИЯ С ЯДРОМ ГИЛЬБЕРТА

На основе метода дискретных особенностей построена дискретная математическая модель сингулярного интегрального уравнения первого рода с ядром Гильберта в случае, когда дополнительное условие, позволяющее получить единственное решение этого уравнения, есть функционал. Доказана однозначная разрешимость дискретной модели и дана оценка скорости сходимости решения дискретной задачи к точному решению сингулярного интегрального уравнения при некоторых предположениях гладкости.

Ключевые слова: сингулярное интегральное уравнение, метод дискретных особенностей, ядро Гильберта, дискретная модель.

Introduction. *Singular integral equations* arise frequently when solving applied physical, mechanical, and engineering problems. In particular, the problem on the distribution of the surface-current density over a narrow circular strip antenna is reduced to a singular integral equation with *the Hilbert kernel* [1]. In this paper we study a singular integral equation with the Hilbert kernel in the case when the auxiliary condition ensuring the existence and uniqueness of the solution is given by a functional. We develop the mathematical framework of a numerical method for solving this equation, which is based on the well-known *discrete singularities method* [2, 3, 4].

Characteristic singular integral equation (SIE) with Hilbert kernel. Let $L^2_{[0,2\pi]}$ be *the Hilbert space* of $2\pi-$periodic functions endowed with the scalar product $(u,v)=\int\limits_0^{2\pi} u(\varphi)\overline{v}(\varphi)d\varphi;$ denote $L^{2,0}_{[0,2\pi]}$ the subspace of $L^2_{[0,2\pi]}$ consisting of its elements that are orthogonal to the unity, i.e. satisfy the equality: $\int\limits_0^{2\pi} u(\varphi)d\varphi=0.$

We introduce the operator

$$(Hu)(\theta)\equiv\frac{1}{2\pi}\int\limits_0^{2\pi}\mathrm{ctg}\frac{\varphi-\theta}{2}u(\varphi)d\varphi\,.$$

The integral in the right-hand part of the identity is singular with the Hilbert kernel. *The Cauchy principal value* of this integral can be computed for any $2\pi-$periodic function $u(\varphi)\in L^2_{[0,2\pi]}$. In case the function $u(\varphi)$ is *Holder continuous*, then $(Hu)(\theta)$ is Holder continuous as well. The operator H takes $L^2_{[0,2\pi]}$ into $L^{2,0}_{[0,2\pi]}$, it is bounded and its norm equals one [2, 3, 4].

Consider the characteristic SIE for the unknown function $u(\varphi)\in L^2_{[0,2\pi]}$:

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\mathrm{ctg}\frac{\varphi-\theta}{2}u(\varphi)d\varphi=f(\theta)\,. \qquad (1)$$

In (1) $f(\theta)\in C^{\mu,\gamma}$ is a given $2\pi-$periodic function; $C^{\mu,\gamma}$ stands for the class of $\mu-$times continuously differentiable functions which μ – th derivatives satisfy the Holder condition with the exponent γ $(0<\gamma\le 1)$.

From the above it follows that for the characteristic SIE to have a solution it is necessary that $f(\theta)\in L^{2,0}_{[0,2\pi]}$, i.e. $\int\limits_0^{2\pi} f(\theta)d\theta=0.$ Moreover, without any auxiliary assumptions imposed on the function $u(\varphi)$ it is impossible to prove the uniqueness of this solution [2, 4]. Sometimes, when solving problems arising in the theory of diffraction of waves the auxiliary assumption ensuring the uniqueness of the solution is introduced in the form of the functional:

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\left\{\ln\left|\sin\frac{\varphi-\alpha}{2}\right|+G(\varphi)\right\}u(\varphi)d\varphi=C\,, \qquad (2)$$

where α and C are given constants, $G(\varphi)$ is a given sufficiently smooth 2π – periodic function such that

$$h\equiv\frac{1}{2\pi}\int\limits_0^{2\pi}\left\{\ln\left|\sin\frac{\varphi-\alpha}{2}\right|+G(\varphi)\right\}d\varphi\ne 0\,. \qquad (3)$$

After introducing the linear substitution $u(\varphi)=w(\varphi)+\dfrac{C}{h}$, condition (2) becomes homogeneous. Since the operator H cancels the constant, equation (1) preserves under this substitution. The aforesaid enables us to study equation (1) supplemented by the following homogeneous condition:

$$\hat{G}u\equiv\frac{1}{2\pi}\int\limits_0^{2\pi}\left\{\ln\left|\sin\frac{\varphi-\alpha}{2}\right|+G(\varphi)\right\}u(\varphi)d\varphi=0\,. \qquad (4)$$

Denote by $\Lambda(G)$ the subspace of $L^2_{[0,2\pi]}$ which elements satisfy condition (4). If limited to the pair of spaces

$$(\Lambda(G),\, L^{2,0}_{[0,2\pi]})\,, \qquad (5)$$

the operator H has a bounded inverse. Hence, problem (1), (4) admits a unique solution for any right-hand part $f(\theta)\in L^{2,0}_{[0,2\pi]}$ [3].

Problem setting for a complete SIE with the Hilbert kernel.

We are looking for a solution $u(\varphi)$ of the complete SIE with Hilbert kernel:

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\mathrm{ctg}\frac{\varphi-\theta}{2}u(\varphi)d\varphi+\frac{1}{2\pi}\int\limits_0^{2\pi}K(\theta,\varphi)u(\varphi)d\varphi=f(\theta)\,, \qquad (6)$$

where $f(\theta)$ and $K(\theta,\varphi)$ are given functions 2π – periodic in θ and φ, $f(\theta)\in C^{\mu,\gamma}$, $K(\theta,\varphi)\in C^{\mu,\gamma}$ in each variable uniformly with respect to the other variable. Moreover, in physical applications the functions $f(\theta)$ and $K(\theta,\varphi)$ satisfy the properties:

$$\int\limits_0^{2\pi} f(\theta)d\theta=0,\qquad \int\limits_0^{2\pi}K(\theta,\varphi)d\theta\equiv 0. \qquad (7)$$

We assume that equation (6) admits a unique solution, which satisfies auxiliary condition (4), and that (3) holds. We also choose $G(\varphi)\in C^{\mu,\gamma}$ in (3).

The necessary condition for equation (6) to have a solution is

$$\int\limits_0^{2\pi}\left\{f(\theta)-\frac{1}{2\pi}\int\limits_0^{2\pi}K(\theta,\varphi)u(\varphi)d\varphi\right\}d\theta=0\,, \qquad (8)$$

where $u(\varphi)$ is the solution to the equation. We arrive at condition (8) by taking $\dfrac{1}{2\pi}\int\limits_0^{2\pi}K(\theta,\varphi)u(\varphi)d\varphi$ to the right-hand part of equation (6) and assuming that the difference $f(\theta)-\dfrac{1}{2\pi}\int\limits_0^{2\pi}K(\theta,\varphi)u(\varphi)d\varphi$ belongs to $L^{2,0}_{[0,2\pi]}$. Properties (7) imply that condition (8) holds for any function $u(\varphi)$ and provides no additional restriction on the solutions to the equation.

We introduce the operator K: $(Ku)(\theta)\equiv\dfrac{1}{2\pi}\int\limits_0^{2\pi}K(\theta,\varphi)u(\varphi)d\varphi$. From (7) it follows that $(Ku)(\theta)\in L^{2,0}_{[0,2\pi]}$. Hence, the operator $H+K$ takes $L^2_{[0,2\pi]}$ into $L^{2,0}_{[0,2\pi]}$. Moreover, if restricted to the couple of spaces (5), this operator admits a bounded inverse. Indeed, since the operator H is continuously invertible in the couple of spaces (5), its index in this couple of spaces equals zero and it does not change after adding the compact operator K to H [5]. Thus the index of the operator $H+K$ restricted to the couple of spaces (5) is zero. Since for a given right-hand part $f(\theta)\in L^{2,0}_{[0,2\pi]}$ equation (6) has a unique solution from $\Lambda(G)$, the restriction of the operator $H+K$ to the couple of spaces (5) possesses a unique bounded inverse operator $(H+K)^{-1}$, which means that problem (6), (4) admits a unique solution for any right-hand part $f(\theta)\in L^{2,0}_{[0,2\pi]}$.

Quadrature formulae and discretization of complete SIE. Let $S_n(\varphi)$ denote a trigonometric polynomial of degree n. We introduce the system of points on a unit circle:

$$\varphi_k^{(1,n)}=\frac{2k\pi}{2n+1},\ \ k=0,1,\ldots,2n;\quad \varphi_k^{(2,n)}=\frac{(2k+1)\pi}{2n+1},\ \ k=0,1,\ldots,2n.$$

Below we list several useful properties of $S_n(\varphi)$ [2, 4]:

$$1.\ \frac{1}{2\pi}\int_0^{2\pi}S_n(\varphi)d\varphi=\frac{1}{2n+1}\sum_{k=0}^{2n}S_n(\varphi_k^{(i,n)}),\ \ i=1,2. \tag{9}$$

Note, that formula (9) holds for any trigonometric polynomials of the degree less than or equal to $2n$ [2].

$$2.\ \frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\varphi-\varphi_j^{(2,n)}}{2}S_n(\varphi)d\varphi=\frac{1}{2n+1}\sum_{k=0}^{2n}\operatorname{ctg}\frac{\varphi_k^{(1,n)}-\varphi_j^{(2,n)}}{2}S_n(\varphi_k^{(1,n)}),\ \ j=0,1,\ldots,2n. \tag{10}$$

$$3.\ \frac{1}{2\pi}\int_0^{2\pi}S_n(\varphi)\ln\left|\sin\frac{\varphi-\varphi_0}{2}\right|d\varphi=-\frac{1}{2n+1}\sum_{k=0}^{2n}S_n(\varphi_k^{(1,n)})\left\{\ln 2+\sum_{m=1}^{n}\frac{1}{m}\cos\left[m\left(\varphi_k^{(1,n)}-\varphi_0\right)\right]\right\}. \tag{11}$$

Denote by $\left(P_n^{(i)}g\right)(\varphi)$ the trigonometric interpolation polynomial of a continuous $2\pi-$periodic function $g(\varphi)$ of degree n with the interpolation nodes $\varphi_k^{(i,n)},\ k=0,1,\ldots,2n;\ i=1,2$.

We are looking for the solution $u_n(\varphi)$ to problem (6), (4) in the form of a trigonometric polynomial of degree n. In most cases, when simply replacing the functions $f(\theta)$ and $K(\theta,\varphi)$ by their interpolation polynomials $\left(P_n^{(2)}f\right)(\theta)$ and $\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)$ and substituting $u_n(\varphi)$ instead of $u(\varphi)$ in (6), the resulting approximate SIE is unsolvable. The cause of this problem is that, with rare exceptions, both $\int_0^{2\pi}\left(P_n^{(2)}f\right)(\theta)d\theta\neq 0$ and $\int_0^{2\pi}\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)d\theta\neq 0$ and the necessary condition for the SIE to have a solution does not hold. That is why we search for $u_n(\varphi)$ as a solution to the following regularized equation:

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\varphi-\theta}{2}u_n(\varphi)d\varphi+\frac{1}{2\pi}\int_0^{2\pi}\left\{\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)-\frac{1}{2\pi}\int_0^{2\pi}\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)d\theta\right\}u_n(\varphi)d\varphi=$$

$$=\left(P_n^{(2)}f\right)(\theta)-\frac{1}{2\pi}\int_0^{2\pi}\left(P_n^{(2)}f\right)(\theta)d\theta, \tag{12}$$

supplemented by the condition:

$$\hat{G}_n u_n\equiv\frac{1}{2\pi}\int_0^{2\pi}\left\{\ln\left|\sin\frac{\varphi-\alpha}{2}\right|+\left(P_n^{(1)}G\right)(\varphi)\right\}u_n(\varphi)d\varphi=0. \tag{13}$$

For problem (12), (13) the necessary condition of the existence of a solution obviously holds.

Condition (3) for problem (12), (13) becomes

$$h_n=\frac{1}{2\pi}\int_0^{2\pi}\left\{\ln\left|\sin\frac{\varphi-\alpha}{2}\right|+\left(P_n^{(1)}G\right)(\varphi)\right\}\neq 0.$$

It is easy to verify that for a smooth function $G(\varphi)$ there exist $h^0>0$ and $N>0$ such that $|h_n|>h^0$ for $n>N$. Below we assume $n>N$.

To construct the system of linear algebraic equations (SLAE) approximating problem (6), (4) it is necessary to introduce an additional (regularizing) unknown by the formula:

$$\beta_n=\frac{1}{2\pi}\int_0^{2\pi}\left\{\left(P_n^{(2)}f\right)(\theta)-\frac{1}{2\pi}\int_0^{2\pi}\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)u_n(\varphi)d\varphi\right\}d\theta. \tag{14}$$

Then equation (12) takes the form:

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\varphi-\theta}{2}u_n(\varphi)\,d\varphi+\frac{1}{2\pi}\int_0^{2\pi}\left\{\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)\right\}u_n(\varphi)\,d\varphi+\beta_n=\left(P_n^{(2)}f\right)(\theta)\,, \tag{15}$$

and the necessary condition for its solvability becomes:

$$\frac{1}{2\pi}\int_0^{2\pi}\left\{\left(P_n^{(2)}f\right)(\theta)-\beta_n-\frac{1}{2\pi}\int_0^{2\pi}\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)u_n(\varphi)\,d\varphi\right\}d\theta=0\,. \tag{16}$$

Clearly, any solution $u_n(\varphi)$ to (12) also solves (15) with β_n computed by (14). The opposite is also true: if $\{u_n(\varphi),\beta_n\}$ is the solution to (15), than it satisfies condition (16) and $u_n(\varphi)$ solves (12). In this sense equations (12) and (15) are equivalent.

We are now in position to construct the system of linear algebraic equations (SLAE) equivalent to equation (15) supplemented by condition (13) by the method of discrete singularities. Since both left and right-hand parts of equation (15) feature trigonometric polynomials of degree n, this equation is equivalent to the following system of $2n+1$ equations:

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\varphi-\varphi_j^{(2,n)}}{2}u_n(\varphi)\,d\varphi+\frac{1}{2\pi}\int_0^{2\pi}\left\{\left(P_{n_\varphi}^{(1)}K\right)\left(\varphi_j^{(2,n)},\varphi\right)\right\}u_n(\varphi)\,d\varphi+\beta_n=\left(P_n^{(2)}f\right)\left(\varphi_j^{(2,n)}\right),\quad j=0,1,\dots,2n\,. \tag{17}$$

Using quadrature formulae (9), (10), (11) to compute the integrals in system (17) and condition (13), we arrive at the following SLAE with respect to the unknowns $u_n\left(\varphi_k^{(1,n)}\right)$, $k=0,1,\dots,2n$ and β_n:

$$\begin{cases}\dfrac{1}{2n+1}\displaystyle\sum_{k=0}^{2n}\left\{\operatorname{ctg}\frac{\varphi_k^{(1,n)}-\varphi_j^{(2,n)}}{2}+K\left(\varphi_j^{(2,n)},\varphi_k^{(1,n)}\right)\right\}u_n\left(\varphi_k^{(1,n)}\right)+\beta_n=f\left(\varphi_j^{(2,n)}\right),\quad j=0,1,\dots,2n;\\ \dfrac{1}{2n+1}\displaystyle\sum_{k=0}^{2n}\left\{\ln 2+\sum_{m=1}^{n}\frac{1}{m}\cos\left(\varphi_k^{(1,n)}-\alpha\right)-G\left(\varphi_k^{(1,n)}\right)\right\}u_n\left(\varphi_k^{(1,n)}\right)=0.\end{cases} \tag{18}$$

Note that the number of the unknowns of system (18) equals the number of its equations.

From the above discussion it follows that system (18) is equivalent to problem (12), (13), i.e. the solution $u_n(\varphi)$ to problem (12), (13) is represented by the interpolation polynomial which values at the interpolation nodes satisfy system (18).

Below we prove that problem (12), (13) and, hence, system (18) admits a unique solution for n sufficiently large.

Auxiliary spaces and operators. We introduce the following spaces:

– $\Lambda(G_n)$ – the subspace of $L^2_{[0,2\pi]}$ which elements satisfy the condition:

$$\hat{G}_n v\equiv\frac{1}{2\pi}\int_0^{2\pi}\left\{\ln\left|\sin\frac{\varphi-\alpha}{2}\right|+\left(P_n^{(1)}G\right)(\varphi)\right\}v(\varphi)\,d\varphi=0\,;$$

– Π_n – the subspace of $L^2_{[0,2\pi]}$ consisting of all the trigonometric polynomials of degree less than or equal to n;

$$\Lambda_n(G_n)\equiv\Lambda(G_n)\cap\Pi_n\,;\qquad \Pi_n^0\equiv\Pi_n\cap L^{2,0}_{[0,2\pi]}\,.$$

Denote $f_n^R(\theta)=\left(P_n^{(2)}f\right)(\theta)-\frac{1}{2\pi}\int_0^{2\pi}\left(P_n^{(2)}f\right)(\theta)\,d\theta$. We introduce the operator K_n^R:

$$\left(K_n^R u_n\right)(\theta)=\frac{1}{2\pi}\int_0^{2\pi}\left\{\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)-\frac{1}{2\pi}\int_0^{2\pi}\left(P_{n_\varphi}^{(1)}P_{n_\theta}^{(2)}K\right)(\theta,\varphi)\,d\theta\right\}u_n(\varphi)\,d\varphi\in\Pi_n^0\,.$$

Using the above operator notations (12) is reduced to

$$\left(H+K_n^R\right)u_n=f_n^R\,.$$

The operator $H+K_n^R$ maps $\Lambda_n(G_n)$ into Π_n^0. Below we show that for sufficiently large n the operator $H+K_n^R$ restricted to the couple of spaces

$$\left(\Lambda_n(G_n),\,\Pi_n^0\right), \tag{19}$$

admits a continuous inverse, and thereby prove the unique solvability of problem (12), (13) and, hence, SLAE (18). We use the following statement [6]:

Theorem 1. *Let X and Y be linear normed spaces, and $\tilde{X}\subset X$ and $\tilde{Y}\subset Y$ be their finite dimensional subspaces such that* $\dim\tilde{X}=\dim\tilde{Y}$ *. Consider the equations*:

the exact one

$$Tx = y\left(x \in X,\, y \in Y\right)$$

and the approximate one

$$\tilde{T}\tilde{x} = \tilde{y}\left(\tilde{x} \in \tilde{X},\, \tilde{y} \in \tilde{Y}\right),$$

where T *and* $\tilde{T}$ *are linear operators,* $T : X \to Y$, $\tilde{T} : \tilde{X} \to \tilde{Y}$.

If the following conditions hold:

a) the operator T *is continuously invertible in* (X, Y),

b) $p = \left\|T^{-1}\right\|_{Y\to X} \left\|T - \tilde{T}\right\|_{\tilde{X}\to\tilde{Y}} < 1$,

then the approximate equation admits a unique solution $\tilde{x}^* \in \tilde{X}$ *for any right-hand part* $\tilde{y} \in \tilde{Y}$. *Moreover, if* $x^* \in X$ *is the exact solution of the equation* $Tx = y$ *and* $\delta = \|y - \tilde{y}\|_Y$, *then*

$$\left\|x^* - \tilde{x}^*\right\|_X \le \left\|T^{-1}\right\|_{Y\to X} (1-p)^{-1}\left[\delta - p\|y\|_Y\right].$$

However, we can't apply Theorem 1 directly for proving the existence of the operator inverse to $H + K_n^R$ in the couple of spaces (19) since $\Lambda_n(G_n)$ is not a subspace of $\Lambda(G)$.

Proof of SLAE solvability. We first consider an auxiliary problem consisting of equation

$$\left(H+K\right)u^{(n)} = f \,, \tag{20}$$

supplemented by the condition

$$\hat{G}_n u^{(n)} \equiv \frac{1}{2\pi}\int_0^{2\pi}\left\{\ln\left|\sin\frac{\varphi-\alpha}{2}\right| + \left(P_n^{(1)}G\right)(\varphi)\right\}u^{(n)}(\varphi)\,d\varphi = 0\,.$$

which means that $u^{(n)}(\varphi) \in \Lambda(G_n)$.

Lemma 1. There exists $N^* \ge N$ such that for $n > N^*$ the operator $H + K$ admits a continuous inverse in the couple of spaces

$$\left(\Lambda\left(G_n\right),\, L_{[0,2\pi]}^{2,0}\right). \tag{21}$$

The proof of Lemma 1 is based on the estimate $\left|(\hat{G}_n - \hat{G})w\right| \le \dfrac{d}{n^{\mu+\gamma}}\|w\|$, where $w \in L_{[0,2\pi]}^2$, d is a constant depending on G only.

Below we use the notation $(H+K)_n^{-1}$ for the operator inverse to $H + K$ in the couple of spaces (21).

Lemma 2. For any non-zero element $g \in L_{[0,2\pi]}^{2,0}$ there exists $\tilde{N}(g) \ge N^*$ such that if $n > \tilde{N}(g)$ then

$$\left\|(H+K)^{-1}g - (H+K)_n^{-1}g\right\| \le \frac{d}{qn^{\mu+\gamma}}\|g\|\cdot\left\|(H+K)^{-1}\right\|_{L_{[0,2\pi]}^{2,0}\to\Lambda(G)},$$

where $q > 0$ is a constant depending on g, $(H+K)^{-1}$ is the operator inverse to $H + K$ in the couple of spaces (5).

Corollary. There exists a constant $D > 0$ such that $\left\|(H+K)_n^{-1}\right\|_{L_{[0,2\pi]}^{2,0}\to\Lambda(G_n)} \le D$ for all $n > N^*$.

The corollary follows immediately from Lemma 2 and the property [5]: if the sequence of linear bounded operators $\{T_n\}$ converges or is at least bounded for each element of the space, then it is uniformly bounded, i.e. the sequence of the norms $\|T_n\|$ is bounded.

Since $\Lambda_n(G_n) \subset \Lambda(G_n)$, we can now apply the results of Theorem 1 for proving the unique solvability of problem (12), (13).

Arguing as in [2], we derive the following estimates:

$$\left\|(H+K) - (H+K_n^R)\right\|_{\Lambda_n(G_n)\to L_{[0,2\pi]}^{2,0}} = \left\|K - K_n^R\right\|_{\Lambda_n(G_n)\to L_{[0,2\pi]}^{2,0}} \le \frac{B}{n^{\mu+\gamma}}, \quad \left\|f - f_n^R\right\| \le \frac{b}{n^{\mu+\gamma}},$$

where B and b are constants which depend on K and f respectively.

Let now $N^{**} > N^*$ be such that for any $n > N^{**}$ we have

$$p_n \equiv \left\|K - K_n^R\right\|_{\Lambda_n(G_n)\to L_{[0,2\pi]}^{2,0}} \cdot \left\|(H+K)_n^{-1}\right\|_{L_{[0,2\pi]}^{2,0}\to\Lambda(G_n)} \le DB\frac{1}{n^{\mu+\gamma}} < 1\,.$$

Then by Theorem 1, for any $n > N^{**}$ equation (12) admits a unique solution $u_n(\varphi)$ in the couple of spaces (19). Besides, if $u^{(n)}(\varphi)$ is the solution to (20) in the couple of spaces (21), then for $n > N^{**}$ the following estimate holds:

$$\left\| u^{(n)} - u_n \right\| \le \frac{D}{(1-p_n)\, n^{\mu+\gamma}} \left(b + DB \| f \| \right).$$

Finally we get the following result:

Theorem 2. *There exists* $N_1 \ge \max\left\{ N^{**}, \tilde{N}(f) \right\}$ *such that for any* $n > N_1$ *equation (12) admits a unique solution in the couple of spaces (19) (and, hence, SLAE (18) admits a unique solution as well). Moreover, the following estimate holds*:

$$\| u - u_n \| \le \left\| u - u^{(n)} \right\| + \left\| u^{(n)} - u_n \right\| \le \frac{1}{n^{\mu+\gamma}} \left\{ \frac{d}{q} \| f \| \cdot \left\| (H+K)^{-1} \right\|_{L^{2,0}_{[0,2\pi]} \to \Lambda(G)} + \frac{D}{1-p_n} \left(b + DB \| f \| \right) \right\} \equiv \delta_n ,$$

where $u(\varphi)$ *is the solution to equation (6) in the couple of spaces (5),* $\delta_n = O\left(\frac{1}{n^{\mu+\gamma}} \right)$ *for* $n \to \infty$.

Taking into account properties (7) we arrive at the following estimate for the regularizing unknown:

$$|\beta_n| \le \sigma_n ,$$

where $\sigma_n = O\left(\frac{1}{n^{\mu+\gamma}} \right)$ for $n \to \infty$ and depends on $f(\theta)$ and $K(\theta, \varphi)$ only.

Conclusions. We study the numerical solution of a singular integral equation with Hilbert kernel by the method of discrete singularities in the case when the auxiliary condition ensuring the existence of a unique solution to the equation is given by a functional. The system of linear algebraic equaions approximating the singular integral equation considered is constructed. It is proved that this system admits a unique solution under some smoothness conditions imposed on the right-hand part of the singular integral equation and the kernel of its regular part, and some assumptions on the functional. Moreover, the mean rate of convergence of the approximate solution to the exact one is estimated.

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